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Some properties of moduli of smoothness of conformal homeomorphisms

Some estimates for the local and integral moduli of smoothness of arbitrary order for the function realizing conformal mapping of the unit disk onto the simply connected domain bounded by the smooth Jordan curve are considered.

The study of the properties of conformal mappings is important for applications in aerodynamics and hydromechanics because technique of conformal mapping is a useful intermediate step that allows for complicated airfoil flow problems to be solved as problems with simpler geometry.

Let consider in the complex plane a simply connected domain G bounded by a smooth Jordan curve Γ. Let $\tau = \tau(s)$ be the angle between the tangent to Γ and the positive real axis, $s = s(w)$ be the arc length on Γ. Suppose that $w = \varphi(z)$ is a homeomorphism of the closed unit disk $\overline{D} = \{z : |z| \le 1\}$ onto the closure $\overline{G}$ of the domain G, conformal in the open unit disk.

In 1912 O. D. Kellogg proved the theorem in which it had been established that if $\tau = \tau(s)$ satisfies condition

$$|\tau(s_1) - \tau(s_2)| \le C_1 |s_1 - s_2|^{\alpha}, \quad 0 < \alpha < 1,$$

then the derivative $\varphi'(e^{i\theta})$ is continuous in the closed unit disk $\overline{D}$ and satisfies condition

$$|\varphi'(e^{i\theta_1}) - \varphi'(e^{i\theta_2})| \le C_2 |\vartheta_1 - \vartheta_2|^{\alpha}$$

with the same α, where C_1 and C_2 are constants. Afterwards this result was generalized in works by several authors.

Afterwards this result was generalized in works by several authors: S. E. Warshawski, J. L. Geronimus, S. J. Alper, R. N. Kovalchuk, L. I. Kolesnik, P. M. Tamrazov (more detailed see [1], [2], [4] and [5]). Some close problems were investigated by V. A. Danilov, E. P. Dolzenko, E. M. Dynkin, N. A. Shirokov, S. R. Bell and S. G. Krantz, V. V. Andrievskii, V. I. Belyi, B. Oktay, D. M. Israfilov and others (more detailed see [2–4] and [6]).

Certain results in the terms of the uniform curvilinear, arithmetic, local and integral moduli of smoothness of arbitrary order were investigated by author [2–8]. In particular, results for local moduli of smoothness were considered in [2], [5] and [8]) and results for integral moduli of smoothness were considered in [2–4] and [7]).

Let consider the noncentralized local arithmetic modulus of smoothness $\omega_{k,z}(f(z),\delta)$ of order k for the function $w = f(z)$ at the point w on the curve γ, that is

$$\omega_{k,w}((z),\delta)=\sup_{(z_0,\dots,z_k)\in\gamma_{w,\delta}(N)}\left|\left[z_0,\dots,z_k;f,z_0\right]\right|,$$

where $\gamma_{w,\delta}(N)$ is the set of collections $(z_0,\dots,z_k)$ such that curvilinear (with respect to the curve γ) distances between points $z_0,\dots,z_k\in\gamma$ satisfy the condition

$$\rho(z_i,z_{i+1})\Big/\rho(z_j,z_{j+1})\le N \quad (N\in[1,+\infty)),\ \rho(z_i,w)\le\delta\ (i,j=1,\dots,k).$$

Theorem 1 [2]. If the local modulus of smoothness $\omega_{k,s_0}(\tau(s),\delta)$ of order k for the function $\tau(s)$ at the point $w_0=w(s_0)$ on the curve Γ satisfies the condition

$$\omega_{k,s_0}(\tau(s),\delta)=O(\delta^{\alpha})(\delta\to 0),$$

then the local modulus of smoothness $\omega_{k,\theta_0}(\varphi'(e^{i\theta}),\delta)$ of the same order k for the derivative $\varphi'(e^{i\theta})$ of the function $\varphi(z)$ at the point $z_0=e^{i\theta_0}$ on ∂D satisfies the condition

$$\omega_{k,\theta_0}(\varphi'(e^{i\theta}),\delta)=O(\delta^{\alpha})(\delta\to 0).$$

Theorem 2 [5]. If the local modulus of smoothness $\omega_{k,\theta_0}(\varphi'(e^{i\theta}),\delta)$ of order k for the derivative $\varphi'(e^{i\theta})$ of the function $\varphi(z)$ at the point $z_0=e^{i\theta_0}$ on ∂D satisfies the condition

$$\omega_{k,\theta_0}(\varphi'(e^{i\theta}),\delta)=O(\delta^{\alpha})(\delta\to 0),$$

then the local modulus of smoothness of the same order k for the function $\tau(s)$ at the point $w_0=w(s_0)$ on the curve Γ satisfies the condition

$$\omega_{k,s_0}(\tau(s),\delta)=O(\delta^{\alpha})(\delta\to 0).$$

Let consider the integral modulus of smoothness of order k for the function $w=f(z)$ on the curve γ introduced [1] by the formula

$$\widehat{\omega}_k(f(z),\delta)=\left\{\int_{\gamma}\left[\omega_{k,z}(f(z),\delta)\right]^p d\lambda(z)\right\}^{1/p},\ 1\le p<+\infty,\ k\in N,$$

where $\lambda=\lambda(z)$ is the linear Lebesgue's measure on the curve.

Theorem 3. Let the integral modulus of smoothness $\widehat{\omega}_k(\tau(s)),\delta)$ of order k for the derivative for the function $\tau(s)$ of the function $\varphi(z)$ on ∂D satisfy the condition

$$\widehat{\omega}_k(\tau(s),\delta)=O(\omega(\delta))(\delta\to 0),$$

where $\omega(\delta)$ is normal majorant satisfying the condition $\int_0^l \frac{\omega(t)}{t}dt < +\infty$. Then the integral modulus of smoothness of the same order k for the derivative $\varphi'(e^{i\theta})$ of the function on the curve Γ satisfies the condition

$$\widehat{\omega}_k(\varphi'(e^{i\theta})),\delta) = O(\nu(\delta))(\delta \to 0)$$

where $\nu(\delta)$ is some integral majorant [2].

In partial case when integral modulus of smoothness $\widehat{\omega}_k(\tau(s)),\delta)$ of order k for the function $\tau(s)$ satisfies Holder condition $\widehat{\omega}_k(\tau(s)),\delta) = O(\delta^\alpha)(\delta \to 0)$, $0 < \alpha < k$, then integral modulus of smoothness $\widehat{\omega}_k(\varphi'(e^{i\theta})),\delta)$ of the function $\varphi'(e^{i\theta})$ satisfies Holder condition $\widehat{\omega}_k(\varphi'(e^{i\theta})),\delta) = O(\delta^\alpha)(\delta \to 0)$ with the same index α.

So, the local and integral moduli of smoothness of the derivative of conformal homeomorphism between the unit disk and simply connected domain bounded by the smooth Jordan curve satisfy the same condition as the local and integral moduli of smoothness for the tangent angles boundary curve.

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