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Alternative quasiconformal mapping of one class of GAP functions

The specificity of the application of conformal mappings in tomographic methods and algorithms based on the Radon transform is analysed. It is proposed to use discrete Fourier transform algorithms to restore images and determine the coordinates of point sources of radiation as a tool for quasi-conformal mapping of functions subjected to the Radon transform. Thanks to the parametric assignment of functions, a simplified method for calculating derivatives in the expansion in a Taylor series has been developed. The method makes it possible to find the derivative of a function given parametrically, without finding an expression for the direct dependence of the function on the argument. Due to the parametric assignment of functions used in tomography problems, it becomes possible to modify the quasi-conformal mapping. When expanding in a Taylor series, both linear and quadratic terms are taken into account. Numerical calculations show that for a limited scanning sector, the errors of the proposed method are smaller than in the case of a classical quasi-conformal mapping.

Introduction.

In the theory of functions of a complex variable, continuous functions are most often considered as analytic continuations of functions of a real variable [1,2]. Specifying a complex function $w(z)$ of a complex variable $z = x + jy$ is equivalent to specifying two real functions of two real variables: $w(z) = u(x, y) + jv(x, y)$.

While introducing the concept of differentiability of a function of a complex variable, by analogy with the corresponding concept of the theory of functions of a real variable, differences of a fundamental nature arise. In particular, the requirement that a function of a complex variable be differentiable on the set of its values imposes the Cauchy-Riemann condition.

It should be noted here that we use the classical definition of an analytic function, which differs from the applied one usually accepted in the literature by the requirement of continuity of partial derivatives. This implies the sufficiency of the condition for the existence of the first differential, i.e., differentiability of any function of several variables.

Leaving aside the singularities of discontinuous functions for the time being, let us turn to the problem of mapping some ε-neighbourhood of a point z_0 – the argument of the function $w(z)$ – onto a υ-neighbourhood of the point $w_0 = w(z_0)$. The mapping is carried out by an analytical function $w(z)$ with the conservation of angles and the constancy of distances. Such a mapping is conformal. For any conformal mapping, there is some orthogonal grid of curves that transforms into a rectangular Cartesian grid. A typical example of an orthogonal grid of curves is a polar grid. In this case, infinitesimal figures (for example, triangles) with a vertex at a point z_0 are transformed into infinitesimal triangles similar to them with a vertex at a point w_0.

The transformation of plane performed by the analytic function has the following important properties in the ε-neighbourhood of the point z_0, for which the derivative is $w'(z_0) \neq 0$. The vectors of all directions outgoing from this point:

- change (increase or decrease) in length by the same number of times, equal to the modulus w';
- rotate through the same angle equal to the argument w'.

Thus, any figures in an infinitely small area (triangles, rectangles, ellipses, etc.) are transformed into similar ones, i.e. keep their shape. Therefore, such a transformation is called a conformal mapping [3 – 5]. Figures of finite dimensions are distorted, although the angles between the tangents to two curves are preserved (so-called conservatism of angles). In general, the properties of conformal mappings of continuous functions have been studied very well. Conformal mappings are used in electrical and radio engineering, aero- and hydrodynamics, and in other engineering applications.

Unfortunately, we can't say the same about conformal mappings of discontinuous functions. For example, the applied aspects of conformal mappings of such a class of discontinuous functions as discrete signals have not yet been studied enough. These include display errors, computational complexity, etc.

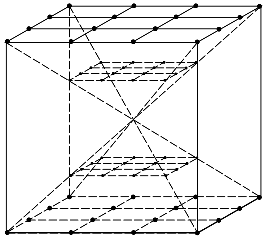
This thesis attempts to fill this gap regard of the problems of tomographic detection and measurement of the coordinates of point sources of acoustic noise radiation [6].

With the reconstructing the tomographic images of large 3D objects from projections and multisite detection/measurement of coordinates of point (small-sized) objects it is sufficient to obtain sets of layered images with the possibility of determining the coordinates of point sources. The problem is to calculate the discrete Radon transform, which is performed on a function with a finite or countable number of discontinuities. Typical examples are functions on Cartesian or polar discrete coordinate grids

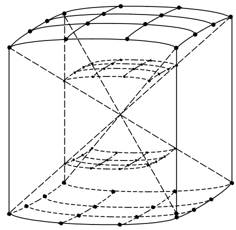
a) Cartesian grid: $x_i = 0, \Delta x, 2\Delta x, \ldots, N\Delta x;\ \ y_i = 0, \Delta y, 2\Delta y, \ldots, N\Delta y.\ \ \Delta x = \Delta y,\ \ N < \infty.$

b) Polar grid: $\varphi_i = 0, \Delta\varphi, 2\Delta\varphi, \ldots, N\Delta\varphi; \quad \rho_i = 0, \Delta\rho, 2\Delta\rho, \ldots, N\,\Delta\rho. \quad N < \infty.$

According to the central section theorem [5], there is a one-to-one correspondence between the Fourier transform and the Radon transform, which, in essence, is the problem of restoring the original function from its integrals along the rays of the polar grid. The properties of the Radon transform and the formulas for its inversion are considered for individual layers, i.e. for the two-dimensional case. Fig. 1 shows layers of three-dimensional functions on coordinate grids.



a) 3D Cartesian grid



b) 3D spherical grid

Fig. 1. Examples of 3D functions with a finite number of discontinuities

Thus, the research problem is formulated as follows:

- to develop a variant of a quasi-conformal mapping of a discontinuous (piecewise-continuous) function;
- to conduct a comparative analysis of the accuracy of various methods for constructing quasi-conformal mappings.

The construction of alternative quasi-conformal mapping.

Let's consider the simplest piecewise-continuous function, which is a collection of points on a circle. This function is typical for the most tomographic problems. Moreover, it's very important problem from the point of view of accuracy of tomographic imaging and detection/measurement.

The points are located at equal angular distances from each other. Fig. 2 shows the sectors of the circle.

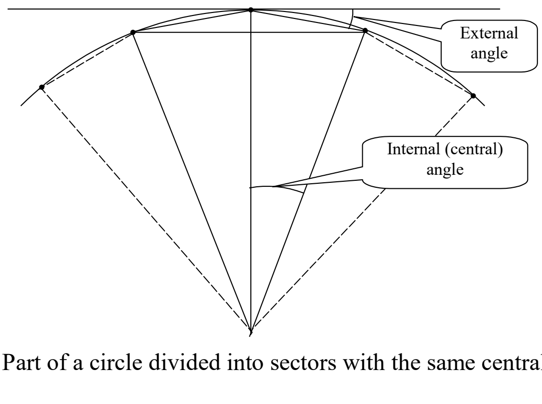


Fig. 2. Part of a circle divided into sectors with the same central angles

Obviously, the smaller the central angle of the sectors, the smaller the conformal mapping errors. Fig. 3 shows graphs of the dependence of internal and external angles, which are of interest for the analysis of conformal mapping errors, the number of partitions of the circle into sectors.

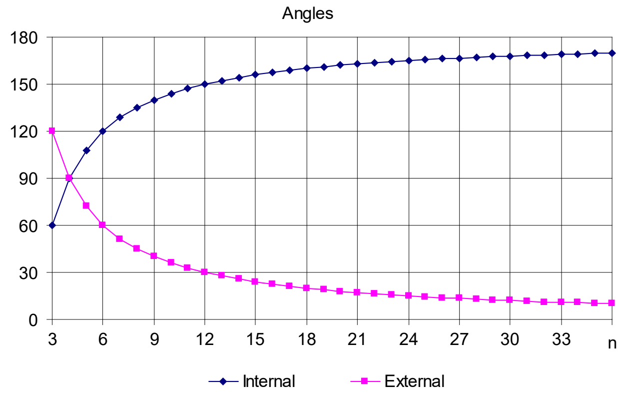


Fig. 3. Graphs of the dependence of internal and external angles on the number of partitions of the circle

Consider the equation of a circle with a radius ρ centered at a point $\{x_0, y_0\}$. We write the equation in parametric form:

$$\begin{cases} x = x_0 + \rho\cos\varphi; \\ y = y_0 + \rho\sin\varphi, \end{cases} \tag{1}$$

where φ is the angle formed by the rotating radius ρ with the positive axis Ox direction.

The function $f(\rho,\varphi)$ has continuous partial derivatives of all orders in the vicinity of the point $\{x_0, y_0\}$ and satisfies the Cauchy-Riemann conditions. Let us write in general terms the expression of the Taylor series for a function of two variables $\{\rho,\varphi\}$:

$$f(\rho,\varphi) = f(\rho_0,\varphi_0) + \frac{1}{1!}\left[\frac{\partial f(\rho_0,\varphi_0)}{\partial\rho}(\rho-\rho_0) + \frac{\partial f(\rho_0,\varphi_0)}{\partial\varphi}(\varphi-\varphi_0)\right] +$$
$$+\frac{1}{2!}\left[\frac{\partial^2 f(\rho_0,\varphi_0)}{\partial\rho^2}(\rho-\rho_0)^2 + 2\frac{\partial^2 f(\rho_0,\varphi_0)}{\partial\rho\partial\varphi}(\rho-\rho_0)(\varphi-\varphi_0) + \frac{\partial^2 f(\rho_0,\varphi_0)}{\partial\varphi^2}(\varphi-\varphi_0)^2\right] + \ldots$$

As applied to the problem under consideration, the circle equation takes the following form:

$$f\left(\varphi_i\big|_{\rho=\text{const}}\right) = \begin{cases} x_0 + \rho\cos\varphi_i; \\ y_0 + \rho\sin\varphi_i, \end{cases} \quad \varphi_i = 0, \Delta\varphi, 2\Delta\varphi, \ldots, n\Delta\varphi \,. \tag{2}$$

Thus, assuming a constant parameter, we pass to a function of one (discrete) variable φ_i. We consider the function $f\left(\varphi_i\big|_{\rho=\text{const}}\right)$ to be discontinuous with the discretisation interval $\varphi_{i+1} - \varphi_i = \Delta\varphi$.

Taking into account the previously introduced assumptions about the properties of the transformation of the complex plane xOy into another complex plane uOv, we expand the functions (2) in a Taylor series with retention of two terms. We call it alternative quasi-conformal mapping.

In order to avoid sophisticated and bulky transformations, we use the parametric definition of function (2). The series will look like this:

$$f\left(\varphi_i\big|_{\rho=\text{const}}\right) = f(\varphi_0) + \frac{1}{1!}\frac{df(\varphi_0)}{d\varphi}(\Delta\varphi - \varphi_0) + \frac{1}{2!}\frac{d^2 f(\varphi_0)}{d\varphi^2}(\Delta\varphi - \varphi_0)^2. \tag{3}$$

Now you need to calculate the first and second derivatives of the function. When defining a function parametrically, you can use universal formulas [7]

$$y'(x) = \frac{y'_\varphi}{x'_\varphi}, \tag{4}$$

$$y''_{xx} = \frac{\left(y'_x\right)'_\varphi}{x'_\varphi} = \frac{\left(\dfrac{y'_\varphi}{x'_\varphi}\right)'}{x'_\varphi} = \frac{y''_\varphi \cdot x'_\varphi - x''_\varphi \cdot y'_\varphi}{\left(x'_\varphi\right)^3}. \tag{5}$$

However, there is an easier way. In order to find the second derivative $y''(x)$ of a given function $\{x = \cos\varphi;\ y = \sin\varphi\}$, we first find its first derivative $y'(x)$:

$$y'_x = \frac{(\sin\varphi)'_\varphi}{(\cos\varphi)'_\varphi} = \frac{\cos\varphi}{-\sin\varphi} = -\operatorname{ctg}\varphi. \tag{6}$$

The second derivative y''_{xx} is formally the first derivative of $y'(x)$, therefore, it can be taken by a formula similar to (4):

$$y''_{xx} = \frac{\left(y'_x\right)'_t}{x'_t} = \frac{(-\operatorname{ctg} t)'_t}{(\cos t)'_t} = \frac{\dfrac{1}{\sin^2 t}}{-\sin t} = -\frac{1}{\sin^3 t}. \tag{7}$$

Specifying in expression (3) the functions of the first and second derivatives (6, 7), we obtain the final expression for the Taylor series with retention of terms up to quadratic inclusive:

$$f\left(\varphi_i\big|_{\rho=\text{const}}\right) = f(\varphi_0) - \operatorname{ctg}\varphi \times (\Delta\varphi - \varphi_0) - \frac{1}{2\sin^3\varphi}(\Delta\varphi - \varphi_0)^2. \tag{8}$$

Calculation of errors of quasi-conformal mapping.

Using expressions (3 – 8), the normalized errors of the quasi-conformal mapping were calculated. Fig. 4 shows graphs of the exact values of the circle equation (2) and approximation by a Taylor series with retention only linear term of expansion. Fig. 5 shows plots of normalized errors of the quasi-conformal mapping during approximation by a Taylor series with retention of only the linear expansion term and with retention of the expansion terms up to including the quadratic one.

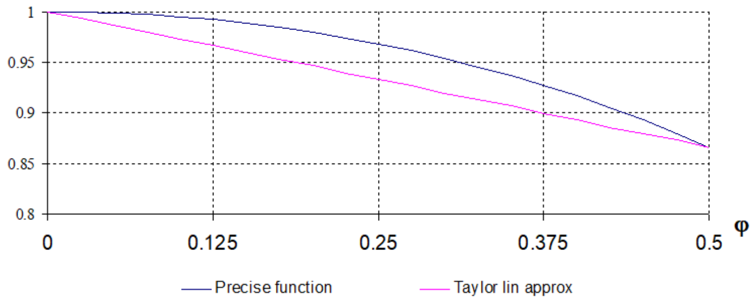


Fig. 4. Graphs of the exact values of the equation and approximation of the Taylor series with a linear expansion term

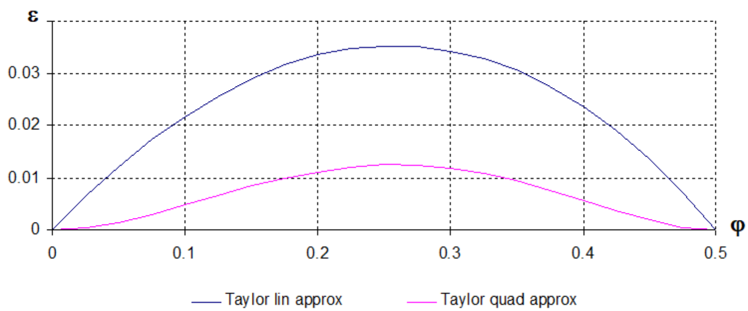


Fig. 5. Plots of normalized errors of a quasi-conformal mapping with one and two expansion terms

The errors of the quasi-conformal mapping depend on the value of the central angle of the sector (and, accordingly, on the number of sectors of the partition of the circle). For example, for a sector with a central angle of 90° (a quadrant), the approximation error will tend to infinity. However, with a reasonable choice of the angular size of the sector, the errors of the quasi-conformal mapping with approximation by the Taylor series with the retention of the terms of the expansion up to the quadratic inclusive will always be smaller. Finally, the using modified quasi-conformal mapping gives us substantial improvement of accuracy.

Conclusion

This study is devoted to the problem of conformal mapping of discontinuous (piecewise-continuous) functions when calculating the inverse Radon transform by successively applying the discrete Fourier transform. For any conformal mapping, there is some orthogonal grid of curves that transforms into a rectangular Cartesian grid. A typical example of an orthogonal grid of curves is a polar grid.

The difficulties that arise when replacing the Radon transform with the Fourier transform can be successfully overcome by applying a quasi-conformal mapping. With fan-shaped scanning, we work in a limited angular range of directions of sighting of a point source of radiation. Then it turns out to be possible to use quasi-conformal mappings and approximation by a Taylor series with retention of terms of the series up to and including the quadratic one.

By applying the parametric assignment of functions that describe the equations of the circular sector, it is possible to significantly simplify the calculation of the derivatives that make up the Taylor series. The comparative analysis of conformal mapping errors and, consequently, the accuracy of calculation results using Fourier transform algorithms is simplified.

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