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About some characteristics of trigonometric splines

The paper investigates the norms in L^2 and seminorms of fundamental trigonometric splines of odd degrees with convergence factors that depend on a parameter; in some cases, such splines coincide with polynomial splines. It is shown that an important characteristic of these splines is their total variation. The material presented is illustrated with graphs.

One of the most successful modifications of algebraic polynomials is polynomial splines, which are sewn from segments of these polynomials according to a certain scheme. The theory of polynomial splines has appeared relatively recently and is well developed (see, for example, [1], [2], [3], etc.). The advantages of polynomial splines include their approximation properties [4].

Subsequently, it turned out [5], [6] that there are also modifications of trigonometric series that depend on several parameters and have the same properties as polynomial splines [7], [8]; moreover, the class of such modified series is quite wide and includes the class of polynomial periodic splines. This gave rise to the name of the class of such series as trigonometric splines.

Important characteristics of trigonometric splines are their norms in the functional space $L^2_{[0,2\pi]}$ [4] and seminorms [1], and, in our opinion, it is also advisable to introduce such a characteristic of these splines as their variation. It is clear that all of these characteristics of the norm are largely determined by the type of chosen convergence factors and the vector Γ, on which the trigonometric splines depend. Therefore, it is relevant to study these characteristics of trigonometric splines with different convergence factors and vectors Γ.

In [8], [11] trigonometric splines with convergence factors $\sigma(\alpha,r,k)=\left(\mathrm{sinc}(\alpha\,\mathrm{k})\right)^{1+r}$ were considered. In the further study of such trigonometric splines, it is important to consider their norms in space L^2 [4] and their seminorms [2]. When comparing the norms and seminorms of trigonometric splines, we chose to compare the norms of fundamental trigonometric splines [11].

In [9] a general method was given for constructing trigonometric fundamental splines $st_k^{(I_1,I_2)}(\Gamma,\sigma,\alpha,r,N,t)$ of degree r, ($r=1,2,...$), with a mesh of stitching I_1, interpolation mesh I_2, parameter vector Γ, and convergence factors $\sigma(\alpha,r,k)$, ($k=1,2,...$), constructed on the meshes $\Delta_N^{(I)}$. If the vector Γ is of the form $\Gamma=\{1,1,1\}$, then the spline is called simple and is denoted by

$st_k^{(I_1,I_2)}(\sigma,\alpha,r,N,t)$. In this article we consider the case where the vector Γ is of the form $\Gamma=\{.1, .5, 1.5\}$; splines with such a vector retain the notation $st_k^{(I_1,I_2)}(\Gamma,\sigma,\alpha,r,N,t)$.

Let us now consider the norms of simple fundamental splines for different powers r, assuming $N=9$ for certainty; recall that the parameter N determines the number of nodes of the mesh $\Delta_N^{(I)}$. The norms of such splines depend on the grid indices I_1, I_2, the vector Γ, the parameter α, which is part of the convergence factor $\sigma(\alpha,r,k)$, the parameter r, which determines the degree of the spline, and the index k ($k=1,2,...,N$) of the grid node. Note that the norms of the splines do not depend on the index of the grid node k.

The graphs of norms and other characteristics of trigonometric splines as functions of the parameter α are given for the case $0<\alpha<\pi/2$; in addition, these graphs are marked with a dotted line to indicate the value of these characteristics of a trigonometric spline that has a polynomial analog. The graphs of the splines themselves are given for $t\in[0,2\pi]$.

Graphs of the norms of splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ and $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ as functions of the parameter α for $r=3,5,7$ are shown in Fig. 1; these norms are denoted by $P0I_2(\alpha,r)$ and $P10I_2(\alpha,r)$, respectively.

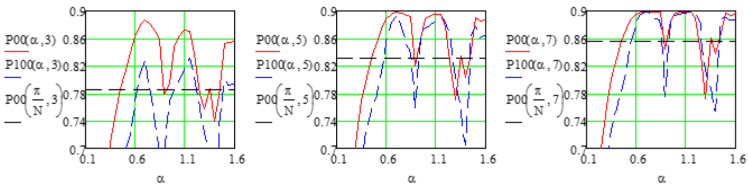

Fig. 1. Graphs of the norms of trigonometric splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ and $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ for $r=3,5,7$

From the analysis of the graphs of the norms of trigonometric splines shown in Fig. 1, it follows that there are values of the parameter α at which the norms of trigonometric splines are less than the norms of polynomial splines.

By comparing the shapes of the graphs of the trigonometric splines obtained at $\alpha=.15$, it is easy to see that at this value of α, undesirable oscillations of the trigonometric splines increase. Thus, it can be concluded that the norms of trigonometric splines do not fully characterize these splines; it is necessary to introduce some other characteristic of trigonometric splines that would numerically characterize the shape of these splines.

There are at least two ways to characterize the shape of trigonometric splines numerically.

In one of them, the presence of oscillations of the splines can be characterized by the variation of these splines on the interval of their change. This approach can be applied to trigonometric splines of both even and odd powers.

The second approach is to study the seminorms of trigonometric splines and apply the consequences of Holladay's theorem [1]. A certain disadvantage of this approach is that it is applicable only to splines of odd degrees. Let us consider both approaches in more detail.

Graphs of variation of trigonometric splines of odd powers are shown in Fig. 2; $V_0^{2\pi}\left[st^{(0,\mathrm{I}_2)}(\sigma,\alpha,r,0,7,t)\right]$ is denoted by $V0I_2(\alpha,r)$ and $V_0^{2\pi}\left[st^{(0,\mathrm{I}_2)}(\Gamma,\sigma,\alpha,r,0,7,t)\right]$ by $V10I_2(\alpha,r)$.

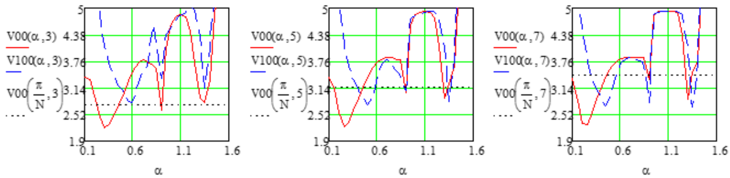

Fig. 2. Graphs of variation of trigonometric splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ and $st^{(0,I_2)}(\Gamma,\sigma,\alpha,r,0,7,t)$ for $r=3,5,7$

We also present (Fig. 3) trigonometric splines with values of the parameter α, at which the smallest variation is achieved (these values are approximate and were chosen from the graphs in Figs. 2); these splines are compared with trigonometric splines of the same powers that have polynomial analogs.

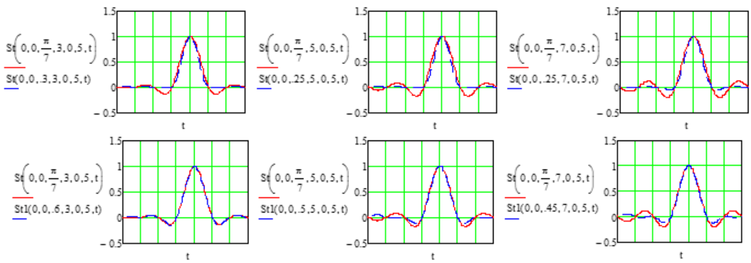

Fig. 3. Graphs of trigonometric splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ and $st^{(0,0)}(\Gamma,\sigma,\alpha,r,0,7,t)$ for $r=3,5,7$

From these graphs it follows that there exist values of the parameter α, at which the trigonometric splines have less variation than the corresponding trigonometric splines of $st^{(0,I_2)}(\sigma,{}^{\pi}\!/\!_{7},2,0,7,t)$, which, as we have already said, have polynomial analogs.

We now present graphs of the seminorms of the odd-degree splines for different values of the parameters α and r (Fig. 4); the seminorms of $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ and $st^{(0,0)}(\Gamma,\sigma,\alpha,r,0,7,t)$ are denoted by $PP00(\alpha,r)$ and $PP100(\alpha,r)$, respectively.

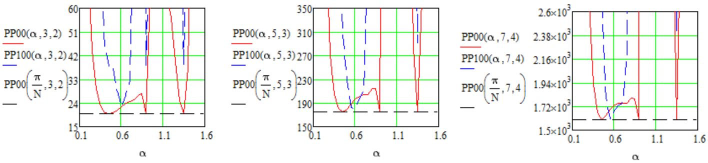

Fig. 4. Graphs of the seminorms of trigonometric splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ and $st^{(0,0)}(\Gamma,\sigma,\alpha,r,0,7,t)$ for $r = 3,5,7$

Analyzing the graphs in Fig. 4, we can draw the following conclusions. First of all, we note that we have obtained an excellent illustration of the property of least curvature of polynomial splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ of odd degree, since they coincide with trigonometric splines of the same degree when $\alpha = {}^{\pi}\!/\!_{N}$.

It also turned out that trigonometric splines have the property of least curvature (and hence coincide with polynomial splines) and for $\alpha = {}^{k\pi}\!/\!_{N}$, ($k$ and N are mutually prime numbers; $k \neq N$, $k = \pm1, \pm2, \ldots$). In other words, there exists a countable set of values of the parameter α at which trigonometric splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ coincide with simple polynomial splines of odd degree. The trigonometric splines $st^{(0,0)}(\Gamma,\sigma,\alpha,r,0,7,t)$ do not have the property of least curvature, and hence, at no value of the parameter α do they coincide with polynomial splines.

Conclusions

It is shown that there exists a countable set of values of the parameter α of the form $\alpha = {}^{k\pi}\!/\!_{N}$, ($|k|$ and N are mutually prime numbers; $|k| \neq N$, $k = \pm1, \pm2, \ldots$), at which simple polynomial splines of odd degree $st^{(0,0)}(\sigma,\alpha,r,0,N,t)$ coincide with polynomial splines.

For a given vector Γ, different types of grids, and values of the parameter r, there exist values of the parameter α at which the variation takes the smallest values.

An illustration of the property of the least curvature of simple polynomial splines is obtained, since trigonometric splines $st^{(0,0)}(\sigma,\alpha,r,0,7,t)$ at values of the parameter $\alpha = {k\pi}/{N}$, (k and N are mutually prime numbers; $k \neq N$, $k = 1,2,\ldots$), coincide with polynomial splines.

Of course, the issues discussed in this paper require further research.

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