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**On some boundary properties of conformal mappings of simply connected domains**

*Some estimates for uniform curvilinear moduli of smoothness of arbitrary order for the function realizing conformal mapping of the unit disk onto the domain bounded by the smooth Jordan curve and for the function realizing conformal mapping between the domains bounded by the smooth Jordan curves are proposed.*

Conformal mappings of the simply connected domains in the complex plane are useful tools for studying the flow of liquid or gas around a certain profile. Therefore, the study of the properties of conformal mappings is important for aerodynamics and hydromechanics.

Let the finite function $w = f(z)$ be defined on a curve $\gamma \subset C$. Let $(z_0,...,z_k)$ be the collection of the points on the curve $\gamma$ and $[z_0,...,z_k; f, z_0]$ be the finite difference of order $k$ for the function $w = f(z)$.

P. M. Tamrazov $[1]$ defined on rectifiable curves the uniform curvilinear moduli of smoothness of order $k$ for the function $w = f(z)$ as $\omega_{k,N,\gamma}((z),\delta) = \sup\limits_{w\in\gamma} \sup\limits_{(z_0,...,z_k)\in\gamma_{w,\delta}(N)} \left|[z_0,...,z_k; f, z_0]\right|$, where $\gamma_{w,\delta}(N)$ is the set of collections $(z_0,...,z_k)$ such that curvilinear (with respect to the curve $\gamma$) distances between points $z_0,..., z_k \in \gamma$ satisfy the condition $\rho(z_i, z_{i+1}) \big/ \rho(z_j, z_{j+1}) \le N$, $(N \in [1,\infty))$, and $\rho(z_i, w) \le \delta$ $(i, j = 1,...,k)$.

Let consider a simply connected domain $G$ in the complex plane bounded by a smooth Jordan curve $\Gamma$. Let $\tau = \tau(s)$ be the angle between the tangent to $\Gamma$ and the positive real axis, $s = s(w)$ be the arc length on the curve $\Gamma$. Let $w = \varphi(z)$ be a homeomorphism of the closed unit disk $\overline{D} = \{z : |z| \le 1\}$ onto the closure $\overline{G}$ of the domain $G$, conformal in the open unit disk $D$. Let $z = \psi(w)$ be the function inverse to the function $w = \varphi(z)$.

Kellog in 1912 proved the theorem in which it had been established that if $\tau = \tau(s)$ satisfies Holder condition with index $\alpha$, $0 < \alpha < 1$, then the derivative $\varphi'(e^{i\theta})$ of the function $\varphi(z)$ on $\partial D$ satisfies Holder condition with the same index $\alpha$. Afterwards this result was generalized in works by several authors: S.E. Warshawski, J.L. Geronimus, S. J. Alper, R.N. Kovalchuk, L I. Kolesnik.

P. M. Tamrazov $[1],[2]$ obtained solid reinforcement for the modulus of continuity of the function $\varphi(z)$ on $\overline{D}$ . Some close problems were investigated by V.A. Danilov, E.P. Dolzenko, E.M. Dynkin, N. A. Shirokov, S. R. Bell and S. G. Krantz and author (more detailed see $[1]$, $[3]$ and $[7]$).

In particular, results in the terms of the uniform curvilinear and integral moduli of smoothness of arbitrary order were received by author ($[3]-[9]$).

**Theorem 1.** $[3]$ Let modulus of smoothness $\omega_k(\tau(s),\delta)$ of arbitrary order $k$ for the function $\tau(s)$ satisfy the condition $\omega_k(\tau(s),\delta)=O[\omega(\delta)](\delta\to 0)$, where $\omega(\delta)$ is normal majorant satisfying the condition $\int\limits_0^l \omega(t)t^{-1}dt<+\infty$ .

Then the nonzero continuous on $\overline{D}$ derivative $\varphi'(z)$ of the function $\varphi(z)$ exists and the uniform curvilinear modulus of smoothness $\widetilde{\omega}_{k,1,\partial D}(\varphi',\delta)$ of order $k$ for the derivative $\varphi'(z)$ of the function $\varphi(z)$ on $\partial D$ satisfies the condition

$$\widetilde{\omega}_{k,1,\partial D}(\arg\varphi',\delta)=O(\widetilde{\mu}(\delta))(\delta\to 0),\quad \widetilde{\omega}_{k,1,\partial D}(\varphi',\delta)=O(\widetilde{\mu}*(\delta))(\delta\to 0),$$

where

$$\widetilde{\mu}(\delta)=\mu(\delta)+\delta^k\int\limits_\delta^l \frac{\mu(t)}{t^{k+1}}dt\ ,\ \widetilde{\mu}*(\delta)=\mu*(\delta)+\delta^k\int\limits_\delta^l \frac{\mu*(t)}{t^{k+1}}dt\ ,$$

$$\mu(\delta)=\omega_k(\tau(s),\delta)\ +\sum_{j=1}^{k-1}\sum_{r_1=1}^{j-1}\cdots\sum_{r_j=1}^{r_{j-1}-1}\left[\omega_k(\tau(s),\delta)\right]^{\frac{k-r_1}{k}}\times$$

$$\times\int\limits_0^l\cdots\int\limits_0^l \frac{\left[\omega_k(\tau(s),x_j)\right]^{\frac{r_j}{k}}\prod\limits_{i=2}^{j}\left[\omega_k(\tau(s),x_{i-1})\right]^{\frac{r_{i-1}-r_i}{k}}}{\prod\limits_{p=1}^{j}x_p\left(1+\left(x_p/x_{p-1}\right)^{r_p}\right)}dx_1\ldots dx_j\ ,$$

$$\mu*(\delta)=\int\limits_0^l \frac{\omega(x_1)}{x_1\left(1+\left(x_1/\delta\right)^k\right)}dx_1+$$

$$+\sum_{j=1}^{k-1}\sum_{r_1=1}^{j-1}\cdots\sum_{r_j=1}^{r_{j-1}-1}\delta^{k-r_1}\int\limits_0^l\cdots\int\limits_0^l x_{j+1}^{r_{j-1}}\left(1+\int\limits_{x_{j+1}}^{l}\frac{\omega(y)}{y_j^{r_j+1}}dy\right)\left(1+\left(\frac{x_{j+1}}{x_j}\right)^{r_{j+11}}\right)\times$$

$$\times\prod_{p=1}^{j}\left(1+\int_{x_p}^{l}\frac{\omega(t_p)}{t_p^{r_{p-1}-r_{p+1}}}dt_p\right)\left(1+\left(\frac{x_p}{x_{p-1}}\right)^{r_{p-1}}\right)^{-1}x_p^{r_{p-1}-r_p-1}dx_1...dx_{j+1}.$$

In partial case when modulus of smoothness $\omega_k(\tau(s),\delta)$ of order $k$ for the function $\tau(s)$ satisfies Holder condition $\omega_k(\tau(s),\delta)=O(\delta^{\alpha})(\delta\to 0)$, $0<\alpha<k$, then the modulus of smoothness $\widetilde{\omega}_{k,1,\partial D}(\varphi',\delta)$ of the same order $k$ for the derivative $\varphi'(z)$ of the function $\varphi(z)$ on $\partial D$ satisfies the condition $\widetilde{\omega}_{k,1,\partial D}(\varphi',\delta)=O(\delta^{\alpha})(\delta\to 0)$ with the same index $\alpha$.

**Theorem 2.** ([5]). Let modulus of smoothness $\omega_k(\tau(s),\delta)$ of order $k$ $(k\in N)$ for the function $\tau(s)$ satisfy the condition $\omega_k(\tau(s),\delta)=O[\omega(\delta)](\delta\to 0)$, where $\omega(\delta)$ is normal majorant satisfying the condition $\int_0^l\omega(t)t^{-1}dt<+\infty$.

Then the nonzero continuous on $\overline{G}$ derivative $\psi'(w)$ of the function $\psi(w)$ exists satisfying on $\Gamma$ the conditions

$$\widetilde{\omega}_{k,1,\partial D}(\arg\psi',\delta)=O(\widetilde{\eta}(\delta))(\delta\to 0),\quad \widetilde{\omega}_{k,1,\partial D}(\psi',\delta)=O(\widetilde{\eta}*(\delta))(\delta\to 0),$$

where

$$\widetilde{\eta}(\delta)=\widetilde{\mu}(\delta)+\delta^{1-k(k-1)/2}\int_\delta^l\frac{\widetilde{\mu}(y)}{y^{k+1}}dy\left(\delta^k\int_\delta^l\frac{\mu(t)}{t^k}dt\right)^{\frac{k(k+1)/2}{k}-1},$$

$$\widetilde{\eta}*(\delta)=\widetilde{\mu}*(\delta)+\delta^{1-k(k-1)/2}\int_\delta^l\frac{\widetilde{\mu}*(y)}{y^{k+1}}dy\left(\delta^k\int_\delta^l\frac{\widetilde{\mu}(t)}{t^k}dt\right)^{\frac{k(k+1)/2}{k}-1}.$$

In partial case when modulus of smoothness $\omega_k(\tau(s),\delta)$ of order $k$ for the function $\tau(s)$ satisfies Holder condition $\omega_k(\tau(s),\delta)=O(\delta^{\alpha})(\delta\to 0)$, $0<\alpha<k$, then the uniform curvilinear modulus of smoothness $\widetilde{\omega}_{k,1,\partial D}(\psi',\delta)$ of the function $\psi(w)$ satisfies the condition $\widetilde{\omega}_{k,1,\Gamma}(\psi,\delta)=O(\delta^{\alpha})(\delta\to 0)$ with the same index $\alpha$.

Let $G_1$ and $G_2$ be the simply connected domains in the complex plane bounded by the smooth Jordan curves $\Gamma_1$ and $\Gamma_2$. Let $\tau_1(s_1)$ be the angle between the tangent to $\Gamma_1$ and the positive real axis, $s_1(\zeta)$ be the arc length on $\Gamma_1$. Let $\tau_2(s_2)$ be the angle between the tangent to $\Gamma_2$ and the positive real axis, $s_2(w)$ be the arc length on $\Gamma_2$. Let $w=f(\zeta)$ be a homeomorphism of the closure $\overline{G_1}$ of the domain $G_1$ onto the closure $\overline{G_2}$ of the domain $G_2$, conformal in the domain $G_1$.

**Theorem 3.** ($[7]$). Let moduli of smoothness $\omega_k(\tau_1(s_1),\delta)$ and $\omega_k(\tau_2(s_2),\delta)$ of order $k$ $(k \in N)$ for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy Holder condition $\omega_k(\tau_1(s_1),\delta) = O(\delta^\alpha)(\delta \to 0)$ and $\omega_k(\tau_2(s_2),\delta) = O(\delta^\alpha)(\delta \to 0)$ with the same index $\alpha$, $0 < \alpha < k$.

Then integral modulus of smoothness $\omega(f',\delta)$ of the derivative of the function $f(\zeta)$ on $\Gamma_1$ satisfies Holder condition $\omega_k(f',\delta) = O(\delta^\alpha)(\delta \to 0)$ with the same index $\alpha$.

So, conformal homeomorphism between two simply connected domains bounded by the smooth Jordan curves with tangent angles satisfying Holder condition with index $\alpha$ for moduli of smoothness of order $k$ satisfies the same Holder condition.

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