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Gamma distributed time-delayed model of two linearly coupled Goodwin oscillators

We have obtained the new system of differential equations for the interaction of two linearly coupled Goodwin's oscillators with gamma distributed time-delay kernels.

An open economy exponential delay time business cycle Goodwin's model for two regions can be written as a following system of two second order differential equations [1]

$$\begin{aligned} &\varepsilon_1\theta_1 \frac{d^2 y_1(t)}{dt^2} + (\varepsilon_1 + s_1\theta)\frac{dy_1(t)}{dt} + s_1 y_1(t) - A_1(t) = F_1(t), \\ &\varepsilon_2\theta_2 \frac{d^2 y_2(t)}{dt^2} + (\varepsilon_2 + s_2\theta)\frac{dy_2(t)}{dt} + s_2 y_2 - A_2(t) = F_2(t), \end{aligned} \tag{1}$$

where

$$F_1(t) = \varphi_1\left(\frac{dy_1(t)}{dt}\right) + m_1 y_1(t) - m_2 y_2(t),$$

$$F_2(t) = \varphi_2\left(\frac{dy_2(t)}{dt}\right) - m_1 y_1(t) + m_2 y_2(t).$$

Here t denote the time, subscripts $i = 1, 2$ denote the economic regions (or the countries), y_i - regional income, s_i - marginal propensity to save, e_i - adjustment time, $\theta_1 = \theta_2 = \theta$ - delay time, m_i - marginal propensity to import, A_i - regional autonomous investment, $\varphi_i(x)$ - nonlinear accelerator,

$$\varphi_i(0) = 0,\ \varphi_i'(x) \ge 0,\ \varphi_i'(0) = r_i > 0,$$

r_i - acceleration coefficient, φ_{ci} and φ_{fi} - the Hicksian 'ceiling' and 'floor'

$$\lim_{x\to-\infty} \varphi_i(x) = \varphi_{fi},\ \lim_{x\to+\infty} \varphi_i(x) = \varphi_{ci}.$$

We assume zero initial conditions

$$y_i(0) = 0,\ \frac{dy_i}{dt}(0) = 0.$$

The system (1) was obtained by generalizations of the well-known Goodwin model for one region [2].

Miki *et al* [3] proposed the fixed delays form of Goodwin's business cycles interaction for two regions. The model [3] contains two first order neutral delay differential equations

$$\varepsilon_1 \frac{dy_{d1}(t)}{dt} + s_1 y_{d1}(t) - A_1(t) = F_1(t-\theta),$$
$$\varepsilon_2 \frac{dy_{d2}(t)}{dt} + s_2 y_{d2}(t) - A_2(t) = F_2(t-\theta), \quad (2)$$

where subscript d means delayed. We assume zero initial functions

$$y_i(t) = 0,\ \theta \le t \le 0 .$$

It is easy to verify that equations (1) and (2) can be written as the following system of integro-differential equations

$$\varepsilon_1 \frac{dy_1(t)}{dt} + s_1 y_1(t) = I_1(t) - m_1 Y_1(t) + m_2 Y_2(t) + A_1(t),$$
$$\varepsilon_2 \frac{dy_2(t)}{dt} + s_2 y_2(t) = I_2(t) + m_1 Y_1(t) - m_2 Y_2(t) + A_2(t),$$
$$Y_1(t) = \int_{-\infty}^{t} w(t-s,\theta) y_1(s) ds,\quad Y_2(t) = \int_{-\infty}^{t} w(t-s,\theta) y_2(s) ds, \quad (3)$$
$$I_1(t) = \int_{-\infty}^{t} w(t-s,\theta) \varphi_1\left(\frac{dy_1(s)}{dt}\right) ds,$$
$$I_2(t) = \int_{-\infty}^{t} w(t-s,\theta) \varphi_2\left(\frac{dy_2(s)}{dt}\right) ds,$$

where $w(t,\theta)$ is the delay kernel satisfying

$$\int_{\infty}^{t} w(t-s,\theta)\, ds = \int_{0}^{\infty} w(s,\theta)\, ds = 1 .$$

For Eq. (1)

$$w(t,\theta) = \theta^{-1} e^{-\frac{t}{\theta}}, \quad (4)$$

and for Eq. (2)

$$w(t,\theta) = \delta(t-\theta), \quad (5)$$

where $\delta(t-\theta)$ is the Dirac delta function.

The most important characteristics of $w(t,\theta)$ are the average delay time T_d, its variance σ_d^2 and coefficient of variation V_d

$$T_d = \int_0^\infty s w(s,\theta)\,ds,\ \ \sigma_d^2 = \int_0^\infty (s-T_d)^2 w(s,\theta)\,ds,\ \ V_d = \frac{\sqrt{\sigma_d^2}}{T_d}.$$

For delay kernel (4) $T_d = \theta$, $\sigma_d = \theta$ and $V_d = 1$, and for kernel (5) $T_d = \theta$, $\sigma_d = 0$ and $V_d = 0$.

The distributions (4) and (5) have significantly different coefficients of variation (1 and 0, respectively). For modeling the more real case

$$0 < V_d < 1$$

can be used the gamma distribution,

$$w_k(s,\theta) = \frac{e^{-\frac{ks}{\theta}}}{\theta(k-1)!}\left(\frac{ks}{\theta}\right)^k,\ \ k = 1,2,... \tag{6}$$

For gamma distribution

$$T_d = \frac{(k+1)}{k}\theta,\ \ \sigma_d^2 = \frac{(k+1)}{k^2}\theta^2,\ \ V_d = \frac{1}{\sqrt{k+1}}.$$

To simulate the time behavior of income for a single Goodwin equation, this distribution was used in [4, 5]. If $k \to \infty$, then Gamma distributions tends to $\delta(s-\theta)$ and Eq. (3) reduces to Eq. (2).

It can be shown that Eqs. (3) are equivalent to the system of ODE's. To see this, we consider such an integral

$$J(t) = \int_{-\infty}^{t} w_k(t-s,\theta) f(s)\,ds \tag{7}$$

We assume that for $s < 0$ $f(s) = 0$ and apply the Laplace transform to Eq. (7). According to the convolution theorem

$$\tilde{J}(p) = \tilde{f}(p)\tilde{W}(p) \tag{8}$$

Since the Laplace transform of the function $w_k(s,\theta)$ is given by

$$\tilde{W}(p) = \frac{1}{\left(1+\frac{\theta}{k}p\right)^{k+1}},$$

then from Eq. (8) we find

$$\left(1+\frac{\theta}{k}p\right)^{k+1}\tilde{J}(p) = \tilde{f}(p). \tag{9}$$

This means that the following differential equation holds

$$\left(1+\frac{\theta}{k}\frac{d}{dt}\right)^{k+1} J(t) = f(t). \tag{10}$$

Therefore Eqs. (3) are equivalent to the system of 4*k*+6 ODE's

$$\varepsilon_1 \frac{dy_1(t)}{dt} + s_1 y_1(t) = I_1(t) - m_1 Y_1(t) + m_2 Y_2(t) + A_1(t),$$

$$\varepsilon_2 \frac{dy_2(t)}{dt} + s_2 y_2(t) = I_2(t) + m_1 Y_1(t) - m_2 Y_2(t) + A_2(t),$$

$$LY_1(t) = y_1(t), \qquad LY_2(t) = y_2(t),$$

$$LI_1(t) = \varphi_1\left(\frac{dy_1(t)}{dt}\right), \quad LI_2(t) = \varphi_2\left(\frac{dy_2(t)}{dt}\right),$$

where

$$L = \left(1 + \frac{\theta}{k}\frac{d}{dt}\right)^{k+1} .$$

Conclusions

To improve the modeling of the time behavior of the income for two linearly coupled Goodwin equations, it is proposed to use a continuous delay model with the kernel in form of a gamma distribution (6). A new system of differential equations is obtained.

References

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