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Crack length distribution model for fatigue damage

This paper proposes the crack lengths distribution model in case of fatigue damage. It is based on experimental studies of the initiation and growth of fatigue cracks in a flat specimen with many holes. Hyperbolic functions and their approximation by Pareto distribution are used to create the model.

The correlation between the number and size of defects is a fundamental characteristic of solids damage in fatigue destruction. Such correlation in the form of statistical distribution of crack lengths can be used to create models of destruction for solids and to solve many problems in prediction of aviation structures bearing capacity. To create the model we will use the data of our own experimental studies. The experiment investigated the propagation of fatigue cracks in the flat specimens with many concentrators at three load levels. The geometry of specimens, the method of fatigue tests conducting, monitoring of the cracks behavior and measuring their lengths are agree with the work [1].

The maximum length of the monitored cracks was limited to the distance between adjacent holes in the specimen and equaled 16 mm. The dependence of the crack lengths *a* on the number of load cycles *N* in this dimensional range can be described in a semi-logarithmic coordinates by the linear function

$$\ln a = p + hN \,. \qquad (1)$$

There were 49 cracks, monitored during the experiment. The growth of each of them was well described by the dependence (1) (Fig. 1). If the initial size of the crack a_0 = 1 mm, from (1) follows

$$p = -hN_0 \,, \qquad (2)$$

where N_0 – the number of cycles to a_0 crack initiation. Substituting the relation (2) into equation (1), we obtain

$$a = \exp[h(N - N_0)] \,. \qquad (3)$$

The coefficient *h* determines the growth rate of the crack on its length and is a function of the operating stress. It should be noted that the exponential growth of fatigue cracks at the initial stage of propagation is characteristic of many materials [2], including aluminum alloys of aircraft structures [3]. At a fixed level of the operating stress, the coefficient *h* of the equation (3) gives the random growth rate of the crack, and therefore it is a random variable. For statistical samples of investigated cracks at operating stresses, the distribution of this coefficient values is satisfactorily described by a uniform law with a distribution density

$$f(h) = \frac{1}{h_{max} - h_{min}} \,, \qquad (4)$$

where h_{min} and h_{max} – interval limits of coefficient *h* possible values.

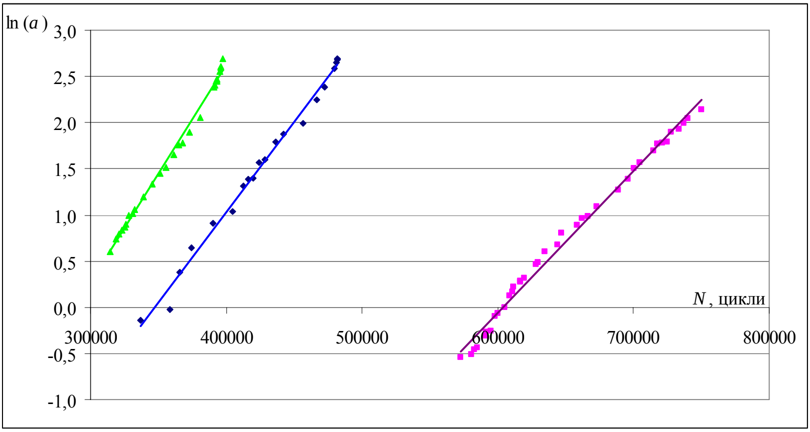

Fig. 1. Approximation of the crack length dependence on the number of cycles by the linear function for stress 90 MPa.

The intensity of the cracks initiation can be determined by the dependences of the accumulated cracks number *n* on the number of load cycles *N* (Fig. 2).

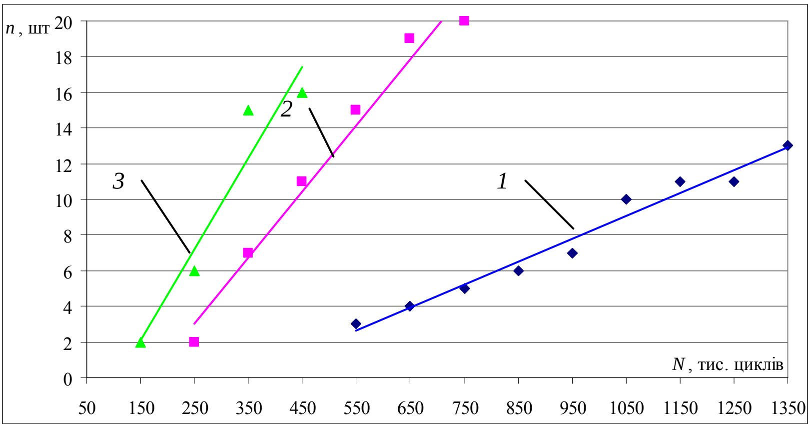

Fig. 2. Dependence of the cracks number in the specimens on the number of load cycles at different values of the maximum stress: *1* – 80 MPa; *2* – 90 MPa; *3* – 110 MPa.

With the linear approximation of the obtained data (Fig. 2), it can be assumed that the intensity of crack formation л for each level of stress is a constant value. The values of the parameter л and its application intervals are summarized in Table 1.

Table 1

The values of the crack initiation intensity and the intervals at different maximum stresses

$у_{max}$, MPa	л , cycle^{-1}	N_{min} , cycle	N_{max} , cycle	Correlation coefficient R^2
80	0,0128	500000	1350000	0,973
90	0,0371	220000	710000	0,976
110	0,0510	100000	360000	0,924

The statistical scatter of the crack length values at a constant level of cyclic stress depends on two random factors – growth and initiation of flaws in time. Assuming that all cracks grow at a deterministic rate, but each has initiated at a random number of cycles, the scatter of their size values would depend only on the duration of growth. The cracks that have initiated earlier would have a greater length. The distribution of the flaw sizes in this case is determined by the distribution of life to crack initiation [4].

Let's consider how the distribution of the crack lengths is described, taking into account their experimentally established regularities of behavior and random initiation and growth. To achieve this goal we will use the approach described in [5].

We introduce a parameter of size *y*, which is related to the crack length as $y = \ln a$. Then (1) can be rewritten in the form

$$y = p + hN . \tag{5}$$

Let's define the distribution function of the parameter *y* at a fixed moment of life $N' > N_{min}$. It is quite obvious that in case of the same growth rate for all cracks, the appearance of some $y < y'$ is possible only if this crack has initiated after y' (i.e. $N > N_0'$). Assuming that the initiation of cracks corresponds to the Poisson flow of events, which has properties of regularity and absence of consequences, the probability of crack initiation in the interval of life $(N_0', N']$ taking into account (3) will be determined as

$$P\left\{\tilde{N}_0 \in (N_0', N'] \middle| h\right\} = 1 - \exp[-л(N' - N_0')] . \tag{6}$$

In accordance with equations (2) and (5) we can write:

$$N' - N_0' = \frac{y'}{h} . \tag{7}$$

Then, taking into account (7), the expression (6) will look like

$$F(y|h) = 1 - \exp\left(-\frac{лy}{h}\right) . \tag{8}$$

From (8) it follows that the conditional distribution function of the parameter *y* for a given value of *h* does not depend on the cyclic load and takes into account only the random initiation of defects. The effect of cracks random growth on the distribution of their lengths can be taken into account by the random variable *h*. Applying the formula of full probability for the conditional distribution function (8) and taking into account the empirical distribution of the parameter *h* (4), we obtain an unconditional distribution function of the flaw sizes parameter *y*:

$$G(y) = 1 - \frac{1}{h_{\max} - h_{\min}} \int_{h\min}^{h\max} \exp\left(-\frac{\text{л}y}{h}\right) dh \tag{9}$$

Applying to (9) integration with the change of variable we have:

$$G(y) = 1 - \frac{1}{h_{\max} - h_{\min}} \left\{ h_{\max} \exp\left(-\frac{\text{л}y}{h_{\max}}\right) - h_{\min} \exp\left(-\frac{\text{л}y}{h_{\min}}\right) + \lambda y \left[E_1\left(\frac{\text{л}y}{h_{\max}}\right) - E_1\left(\frac{\text{л}y}{h_{\min}}\right) \right] \right\} . \tag{10}$$

The formula for the distribution density of the parameter *y* is obtained by differentiating (10) by this parameter:

$$g(y) = \frac{\text{л}}{h_{\max} - h_{\min}} \left[E_1\left(\frac{\text{л}y}{h_{\max}}\right) - E_1\left(\frac{\text{л}y}{h_{\min}}\right) \right], \tag{11}$$

The change of the parameter *y* distribution (11) into the distribution of the crack lengths *a* is performed on the basis of the variate transformation rule and we obtain the formula for the crack length density:

$$f(a) = \frac{\text{л}}{a(h_{\max} - h_{\min})} \left[E_1\left(\frac{\text{л} \ln a}{h_{\max}}\right) - E_1\left(\frac{\text{л} \ln a}{h_{\min}}\right) \right]. \tag{12}$$

Note that the function (12) is positive and corresponds to normalization condition $\int_1^{\infty} f(a)da = 1$, that meets the requirements for the distribution density of a random variable. The calculations carried out by the formula (12), taking into account the experimental values of the parameters л, $h_{\min}$,and $h_{\max}$ indicate the hyperbolic type of function for the density distribution of the crack length *a*. The calculated dependencies are well described by the functions of the form:

$$\varphi(a) = \frac{\text{г} - 1}{a^{\text{г}}}. \tag{13}$$

For $\text{г} > 1$ and bounded interval *a* ($a \geq 1$ mm), the function (13) has the property of a one-parameter distribution density of the crack lengths – it is positive and satisfies the condition of normalization. The function of the crack length distribution corresponding to the density (13) in the $a \geq 1$ mm region is determined as

$$F(a) = 1 - \frac{1}{a^{\text{г}-1}}. \tag{14}$$

Consequently, instead of formula (12), a simpler expression (13) can be used to describe the crack length distribution. In the further analysis of the dependencies (13) and (14) it was found that they are converted into a well-known Pareto distribution by a simple substitution $a_0 = 1$ and $k = \text{г} - 1$. Pareto's distribution function is $F(a) = 1 - \left(\frac{a_0}{a}\right)^k$ and distribution density – $f(a) = \frac{k a_0^k}{a^{k+1}}$. Calculations carried out on the given formulas well approximate experimentally obtained density distributions of the crack lengths (Fig. 3).

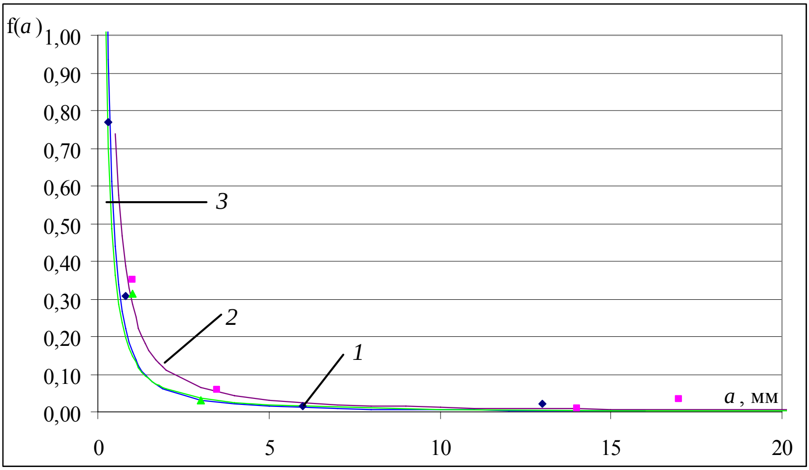

Fig. 3. Approximation of the experimental distribution densities of the crack lengths at different values of the maximum stress: *1* – 80 MPa; *2* – 90 MPa; *3* – 110 MPa.

Conclusion. The probabilistic distribution of the fatigue crack length obtained on the basis of their random initiation and growth parameters in case of multiple site damaged specimens with many concentrators is well described by the power function of the hyperbolic type. Taking into account experimentally determined values of that parameters, this function is converted into the distribution density of the crack lengths based on the Pareto distribution, which can be used to describe the general case of multiple destruction of solids.

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