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The Fourier series summation method with $\sigma_k(r,\alpha)$ -factors

The method of Poisson-Abel type of summation of Fourier series, namely, the method of summation with $\sigma_k(r,\alpha)$ - factors is considered in this paper. It is proved in this paper that application of method of summation with $\sigma_k(r,\alpha)$ -factors of Fourier series of periodical function $f(t)$ derives to the convolution of this function with kernels $De(r,\alpha,t)$; if the parameter r is integer, these kernels become polynomial normalized basic B -splines of order $r-1$ ($r=1,2,\ldots$).

Introduction

Let a periodic function $f(t)$ with the period 2π and absolutely integrable on the interval $[-\pi,\pi]$ have its corresponding Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos kt + b_k \sin k\alpha\right]. \tag{1}$$

In many cases a series of this kind is not uniformly convergent. For it to be uniformly convergent, linear methods for summing series are often used. Among such methods of summation, we will consider methods of the Poisson-Abel type. As is known [1], the main idea of Poisson-Abel type methods for summing the series (1) lies in the fact that all terms of the series are multiplied by factors of a certain kind; to put it another way, the series (1) has its corresponding series of the kind:

$$f(\alpha_1,\alpha_2,\ldots,\alpha_n,t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\mu_k(\alpha_1,\alpha_2,\ldots,\alpha_n)\left[a_k \cos kt + b_k \sin k\alpha\right], \tag{2}$$

where $\mu_k(\alpha_1,\alpha_2,\ldots,\alpha_n)$, ($k=1,2,\ldots$) are some factors dependent on n ($n=1,2,\ldots$) parameters $\alpha_1,\alpha_2,\ldots,\alpha_n$; such factors are often called convergence factors.

According to the type of the factors $\mu_k(\alpha_1,\alpha_2,\ldots,\alpha_n)$ concrete methods of summation are distinguished. We will restrict our consideration to the Poisson-Abel method and the method of $\sigma_k(r,\alpha)$ -factors.

The Poisson-Abel method suggests that $\mu_k(\alpha_1,\alpha_2,\ldots,\alpha_n)=r^k$, (i.e. $n=1$, $\alpha_1=r$). According to (2), the function $f(t)$ (1) may have its corresponding series

$$f(r,t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k \left[a_k \cos kt + b_k \sin k\alpha\right], \; (\, 0 < r < 1\,). \qquad (3)$$

Which after few simple transformations you can submit in the form

$$f(r,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k \left[a_k \cos kt + b_k \sin k\alpha\right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u+t) P(r,u)\, du \; . \qquad (4)$$

This integral is referred to as the Poisson integral, and the kernel $P(r,u)$ of the integral transformation

$$P(r,u) = \frac{1}{2} \frac{1-r^2}{1 - 2r\cos u + r^2}$$

as the Poisson kernel. The Poisson integral has been theoretically justified by Schwarz.

A shortcoming of the Poisson-Abel method is that the parameter r has no interpretation. It results in the fact that only the boundary function $f(1-0,t) = \lim\limits_{r \to 1-0} f(r,t)$ is taken into consideration when this method is applied. However, in many cases boundary transition is not feasible. It accounts for the fact that attention is drawn to methods in which parameters defining the method could be interpreted in this or that way. When using such methods, we can consider not only boundary functions obtained as a result of boundary transition by some parameters, but also functions derived with these parameters being fixed. One of such methods is the summation method with $\sigma_k(r,\alpha)$ factors, which we consider here.

In the summation method with $\sigma_k(r,\alpha)$ - factors we assume

$$\mu_k(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sigma_k(r,\alpha)\,,$$

where

$$\sigma_k(r,\alpha) = \left(\sin k\frac{\alpha}{2} \Big/ k\frac{\alpha}{2} \right)^r , \quad (\, 0 < \alpha < \pi\, ;\; r = 1,2,\ldots\,). \qquad (5)$$

It is clear that in this case $n = 2$, $\alpha_1 = \alpha$; $\alpha_2 = r$.

The summation method with $\sigma_k(r,\alpha)$ - factors suggests that the series (1) has its corresponding series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \sigma_k(r,\alpha) \left[a_k \cos kt + b_k \sin k\alpha\right], \quad (\, r \geq 1\,) \qquad (6)$$

As in the case before, it is easy to derive the expression

$$f(r,\alpha,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u+t) \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \sigma_k(r,\alpha) \cos ku \right\} du \; . \qquad (7)$$

Introduce the designation

$$De(r,\alpha,t) = \frac{1}{2} + \sum_{k=1}^{\infty} \sigma_k(r,\alpha) \cos k(t) \qquad (8)$$

The kernel $De(r,\alpha,t)$ will be referred to as De-kernel of the r -th order.

By means of immediate verification we can easily see that the kernel $De(r,\alpha,t)$, to the nearest constant factor α, coincides with periodically continued polynomial normalized періодично продовженими поліноміальними нормалізованими B - splines of the order $r-1$ ($r=1,2,...$) with $\alpha=h$, which are formed on uniformly spaced grids with a step h, see [3], [4]. Marking such splines as $B_{r-1}(\alpha,t)$ we get

$$\frac{1}{\alpha}B_{r-1}(\alpha,t)=\frac{1}{\pi}\left(\frac{1}{2}+\sum_{k=1}^{\infty}\sigma_k(r,\alpha)\cos kt\right), \tag{9}$$

where the spline $B_{r-1}(\alpha,t)$ is formed on a uniformly spaced grid with the step α, the symmetry centres of both function being concordant. We will note that with $r=1,\dots,5$ explicit representation of B -splines can be used [5].

It is easy to make sure that the kernels $De(r,\alpha,t)$, for any fixed $r=1,2,\dots$, form a δ -like series [6] with consequently,

$$\lim_{\alpha\to 0} f(r,\alpha,t)=\lim_{\alpha\to 0}\frac{1}{\alpha}\int_{-\pi}^{\pi} f(t+u)B_{r-1}(\alpha,\mathrm{u})\,du=\frac{1}{2}\left[f(t-0)+f(t+0)\right].$$

In particular, in the continuity point this limit equals to $f(t)$.

So, as it took place earlier, we may conclude that the summation method with $\sigma_k(r,\alpha)$ - factors is F - effective [1].

It is well understood that the parameters r and α, which are included in the expression for $\sigma_k(r,\alpha)$ -factors, anticipate easy interpretation. In particular, the parameter r defines the differential features of the kernel $De(r,\alpha,t)$ whereas the parameter α – for functions that have points of discontinuity of the first kind (jump-type discontinuity) – defines the vicinities of these points in which these discontinuities are smoothed.

Conclusions

It is determined that the sum of the Fourier series of an absolutely integrable function $f(t)$ with convergence factors $\gamma_k(\alpha_1,\alpha_2,\dots,\alpha_n)$ introduced into it is equal to the convolution of this function with the kernel $De(t,\alpha_1,\alpha_2,\dots,\alpha_n)$, provided that the series is uniformly convergent, the kernel being determined by the equation

$$De(t,\alpha_1,\alpha_2,\dots,\alpha_n)=\frac{1}{\pi}\left(\frac{1}{2}+\sum_{k=1}^{\infty}\gamma(\alpha_1,\alpha_2,\dots,\alpha_n)\cos kt\right).$$

In particular, it is shown here that the trigonometric Fourier series of an absolutely integrable function $f(t)$, which are summed by the summation method with $\sigma_k(r,\alpha)$ -factors, converge to the convolution of this function with polynomial

B - splines. The establishment of this relationship offers great opportunities to research the classes of polynomial and trigonometric approximations in their entirety.

The factors $\sigma_k(r,\alpha)$ are the Fourier coefficients of normalized basis B - splines of the order $r-1$, ($r=1,2,...$), whereas the kernels $De(r,\alpha,t)$ can be viewed as a trigonometric representation of such splines.

There is an opportunity to use other types of finite functions with a fixed smoothness, in particular spline-type functions, as $De(r,\alpha,t)$ kernels. Moreover, it becomes possible to construct kernels of the $De(r,\alpha,t)$ type with pre-assigned features. The summation factors of the $\sigma_k(r,\alpha)$ type for such kernels can be easily obtained as Fourier coefficients in expansions of these kernels according to (9).

Lastly, one more reason why the trigonometric functions summation method with $\sigma_k(r,\alpha)$ factors attracts attention is that it represents a generalization of the results described in [1,2,7]. For example, A. Zygmund and G. Hardy represent the $De(2,\alpha,t)$ kernel in terms of $\sigma_k(2,\alpha)$ factors, and N. Bari – the $De(3,\alpha,t)$ kernel in terms of $\sigma_k(3,\alpha)$ factors. Nevertheless, these authors have not gone far towards the generalization of these results.

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